

INVERSE SPECTRAL THEORY AND THE MINKOWSKI PROBLEM FOR THE SURFACE OF REVOLUTION

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ABSTRACT. We solve the inverse spectral problem for rotationally symmetric manifolds, which include the class of surfaces of revolution, by giving an analytic isomorphism from the space of spectral data onto the space of functions describing the radius of rotation. An analogue of the Minkowski problem is also solved.

1. INTRODUCTION AND MAIN RESULTS

1.1. The surface of revolution. Suppose we are given a surface of revolution M in \mathbb{R}^{m+2} with $m \geq 1$. Using the coordinates $(x, y) \in \mathbb{R}^{m+2} = \mathbb{R}^1 \times \mathbb{R}^{m+1}$, M is represented as

$$(1.1) \quad y = f(x)\omega, \quad \omega \in \mathbb{S}^m, \quad x \in I = [0, x_0],$$

where $f \in C^2(I)$, $f(x) > 0$. Then the induced metric on M is

$$(1.2) \quad ds^2 = (1 + f'(x)^2) (dx)^2 + f(x)^2 g_{\mathbb{S}^m},$$

$g_{\mathbb{S}^m}$ being the standard metric on \mathbb{S}^m . Making the change of variable $t = t(x)$ by

$$(1.3) \quad \frac{dt}{dx} = \sqrt{1 + f'(x)^2},$$

we can rewrite ds^2 as

$$(1.4) \quad \begin{cases} ds^2 = (dt)^2 + r(t)^2 g_{\mathbb{S}^m}, & r(t) = f(x(t)), \\ 0 \leq t \leq t_0 = \int_0^{x_0} \sqrt{1 + f'(x)^2} dx. \end{cases}$$

Then we have

$$|r'(t)| < 1,$$

since

$$(1.5) \quad r'(t) = f'(x(t)) \frac{dx}{dt} = \frac{f'(x(t))}{\sqrt{1 + f'(x(t))^2}}.$$

Now, the Laplace-Beltrami operator on M is written as

$$(1.6) \quad \Delta_M = \frac{1}{r^m} \partial_t (r^m \partial_t) + \frac{\Delta_Y}{r^2},$$

where Δ_Y is the Laplace-Beltrami operator on \mathbb{S}^m . By imposing suitable boundary conditions on $t = 0$ and $t = t_0$, one can get the spectral data for M . We are

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interested in the inverse spectral problem, i.e. the recovery of M from its spectral data. Note that in this setting, we are given the operator Δ_Y . The value of t_0 is not known a-priori, since it is computed from (1.4), which contains unknown $f(x)$. However, the eigenvalue problem for (1.6) is reduced to the 1-dimensional Sturm-Liouville problem, and one can derive the value of t_0 from the asymptotics of eigenvalues. By virtue of (1.5), (1.3) is rewritten as

$$(1.7) \quad \frac{dx}{dt} = \sqrt{1 - r'(t)^2},$$

from which one can compute $x(t)$ as well as x_0 . We can then recover $f(x)$ from the formula $r(t) = f(x(t))$ and the inverse function theorem.

We have thus seen that our problem is reduced to the inverse spectral problem for (1.6) defined on $[0, t_0] \times Y$ with suitable boundary condition. Since t_0 is known from the spectral asymptotics, we can assume without loss of generality that $t_0 = 1$. More precisely, in the general case, we have only to repeat the arguments below with $M = [0, 1] \times Y$ replaced by $M = [0, x_0] \times Y$, where x_0 is computed from (1.7) and t_0 .

1.2. Rotationally symmetric manifold. Let us slightly generalize our problem. Assume that we are given a compact m -dimensional Riemannian manifold (Y, g_0) (with or without boundary). We consider a cylindrical manifold $M = [0, 1] \times Y$ with warped product metric

$$(1.8) \quad g = (dx)^2 + r^2(x)g_0.$$

The Laplace-Beltrami operator on M is written as

$$(1.9) \quad \Delta_M = \frac{1}{r(x)^m} \partial_x \left(r(x)^m \partial_x \right) + \frac{1}{r^2(x)} \Delta_Y.$$

Two examples are given in Fig. 1, where $Y = \mathbb{S}^1$ and Fig. 2, where $Y = [0, \alpha]$ with a suitable boundary condition on ∂Y .

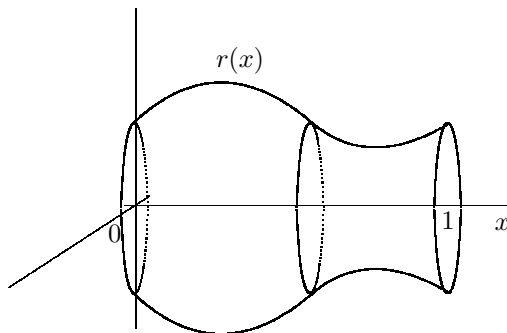
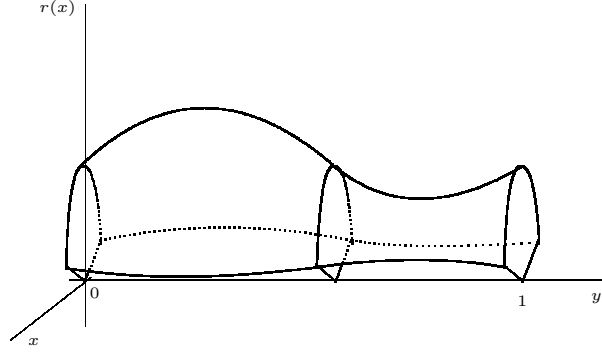


FIGURE 1. The surface with $Y = \{y \in \mathbb{S}^1\}$.


 FIGURE 2. The surface of revolution of an angle $\alpha < \pi$

For the operator (1.9), we impose one of the following boundary conditions on $\partial M = \{0, 1\} \times Y$: For $y \in Y$,

$$(1.10) \quad \begin{cases} \text{Dirichlet b.c.} & f(0, y) = f(1, y) = 0, \\ \text{Mixed b.c.} & f(0, y) = 0, \quad f'(1, y) + bf(1, y) = 0, \quad b \in \mathbb{R}, \\ \text{Robin b.c.} & f'(0, y) - af(0, y) = 0, \quad f'(1, y) + bf(1, y) = 0, \quad a, b \in \mathbb{R}. \end{cases}$$

The Laplacian $-\Delta_Y$ on Y has the discrete spectrum

$$0 \leq E_1 \leq E_2 \leq E_3 \leq \dots$$

with an associated orthonormal family of eigenfunctions $\Psi_\nu, \nu \geq 1$, in $L^2(Y)$. Then, we have the orthogonal decomposition

$$L^2(M) = \oplus_{\nu \geq 1} \mathcal{L}_\nu^2(M),$$

$$\mathcal{L}_\nu^2(M) = \left\{ h(x, y) = f(x) \Psi_\nu(y); \int_0^1 |f(x)|^2 r^m(x) dx < \infty \right\}, \quad \nu \geq 1.$$

Thus, $-\Delta_M$ is unitarily equivalent to a direct sum of one-dimensional operators,

$$-\Delta_M \simeq \oplus_{\nu=1}^\infty (-\Delta_\nu),$$

$$(1.11) \quad -\Delta_\nu = -\frac{1}{\varrho^2} \partial_x (\varrho^2 \partial_x) + \frac{E_\nu}{r^2}, \quad \varrho = r^{m/2}, \quad \text{on } L^2([0, 1]; r^m(x) dx).$$

We call $-\Delta_\nu$ a Sturm-Liouville operator. The boundary condition (1.10) is inherited for $-\Delta_\nu$:

$$(1.12) \quad \begin{cases} \text{Dirichlet b.c.} & f(0) = f(1) = 0, \\ \text{Mixed b.c.} & f(0) = 0, \quad f'(1) + bf(1) = 0, \quad b \in \mathbb{R}, \\ \text{Robin b.c.} & f'(0) - af(0) = 0, \quad f'(1) + bf(1) = 0, \quad a, b \in \mathbb{R}. \end{cases}$$

The operator $-\Delta_\nu$ actually depends on

$$(1.13) \quad \frac{\varrho'}{\varrho} \quad \text{and} \quad \rho(0) = r(0)^{m/2}.$$

For our purpose, it is convenient to introduce a parameter $q_0 = \rho'(0)/\rho(0)$ and put

$$(1.14) \quad \frac{\rho'(x)}{\rho(x)} = q_0 + q(x).$$

Then $r(x)$ is written as

$$(1.15) \quad r(x) = r(0)e^{2Q(x)/m}, \quad Q(x) = \int_0^x (q_0 + q(t))dt.$$

We then have

$$(1.16) \quad \frac{r'(0)}{r(0)} = \frac{2q_0}{m}, \quad \log \frac{r(1)}{r(0)} = \frac{2}{m} \left(q_0 + \int_0^1 q(x)dx \right).$$

This implies that, if we are given either $r(0)$ and $r'(0)$, or $r(0)$ and $r(1)$, we can reconstruct $r(x)$ from $q(x)$ for $0 \leq x \leq 1$.

The problem we address in this paper is the characterization of the range of the *spectral data mapping*

$$q \rightarrow \{\mu_n(q), \kappa_n(q)\}_{n=1}^\infty,$$

where μ_n and κ_n are eigenvalues and norming constants for (1.11) with a fixed ν .

1.3. Function spaces. Let us introduce the following spaces of real functions

$$(1.17) \quad \begin{aligned} \mathcal{W}_1^0 &= \left\{ q \in L^2(0, 1) ; q' \in L^2(0, 1), q(0) = q(1) = 0 \right\}, \\ \mathcal{H}_\alpha &= \left\{ q \in L^2(0, 1) ; q^{(\alpha)} \in L^2(0, 1), \int_0^1 q^{(j)}(x)dx = 0, \forall j = 0, \dots, \alpha \right\}, \end{aligned}$$

where $\alpha \geq 0$, equipped with norms

$$\|q\|_{\mathcal{W}_1^0}^2 = \|q'\|^2 = \int_0^1 |q'(x)|^2 dx, \quad \|q\|_{\mathcal{H}_\alpha}^2 = \|q^{(\alpha)}\|^2 = \int_0^1 |q^{(\alpha)}(x)|^2 dx.$$

Define the spaces of even functions $L_{even}^2(0, 1)$, and of odd functions $L_{odd}^2(0, 1)$ by

$$(1.18) \quad \begin{aligned} L_{even}^2(0, 1) &= \left\{ q \in L^2(0, 1) ; q(x) = q(1-x), \quad \forall x \in (0, 1) \right\}, \\ L_{odd}^2(0, 1) &= \left\{ q \in L^2(0, 1) ; q(x) = -q(1-x), \quad \forall x \in (0, 1) \right\}, \\ L^2(0, 1) &= L_{even}^2(0, 1) \oplus L_{odd}^2(0, 1) \end{aligned}$$

and for $\omega = even$ or $\omega = odd$ we define

$$(1.19) \quad \mathcal{W}_1^{0, \omega} = \mathcal{W}_1^0 \cap L_\omega^2(0, 1), \quad \mathcal{H}_\alpha^\omega = \mathcal{H}_\alpha \cap L_\omega^2(0, 1), \quad \alpha \geq 0.$$

We also introduce the space ℓ_α^2 of real sequences $h = (h_n)_{n=1}^\infty$, equipped with the norm

$$(1.20) \quad \|h\|_\alpha^2 = 2 \sum_{n \geq 1} (2\pi n)^{2\alpha} |h_n|^2, \quad \alpha \in \mathbb{R},$$

and let $\ell^2 = \ell_0^2$. Finally we define the set $\mathcal{M}_1 \subset \ell^2$ by

$$(1.21) \quad \mathcal{M}_1 = \mathcal{M}_1((\mu_n^0)_{n=1}^\infty) = \left\{ (h_n)_{n=1}^\infty \in \ell^2 ; \mu_1^0 + h_1 < \mu_2^0 + h_2 < \dots \right\},$$

where the sequence $(\mu_n^0)_{n=1}^\infty$ will be specified below.

1.4. Main results I. Spectral data mapping. We are now in a position to stating our main results of this paper.

1.4.1. *Dirichlet boundary condition.* First we consider $-\Delta_\nu, \nu \geq 1$, on the interval $[0, 1]$ with Dirichlet boundary condition:

$$(1.22) \quad \begin{cases} -\Delta_\nu f = -\frac{1}{\varrho^2}(\varrho^2 f')' + \frac{E_\nu}{r^2} f, & \varrho = r^{m/2}, \quad \text{on } (0, 1), \\ f(0) = f(1) = 0. \end{cases}$$

Denote by $\mu_n = \mu_n(q), n = 1, 2, \dots$, the eigenvalues of $-\Delta_\nu$. It is well-known that all μ_n are simple and satisfy

$$(1.23) \quad \begin{aligned} \mu_n &= \mu_n^0 + c_0 + \tilde{\mu}_n, \\ \mu_n^0 &= (n\pi)^2, \quad (\tilde{\mu}_n)_1^\infty \in \ell^2, \\ c_0 &= \int_0^1 \left((q_0 + q)^2 + \frac{E_\nu}{r^2} \right) dx, \end{aligned}$$

where $\mu_n^0, n \geq 1$, are the eigenvalues for the unperturbed case $r = 1$. Following [21], [10], we introduce the norming constants

$$(1.24) \quad \varkappa_n(q) = \log \left| \frac{\varrho(1)f'_n(1, q)}{f'_n(0, q)} \right|, \quad n \geq 1,$$

where f_n is the n -th eigenfunction of $-\Delta_\nu$. Note that $f'_n(0) \neq 0$ and $f'_n(1) \neq 0$. Recall that

$$(1.25) \quad q_0 = \frac{\rho'(0)}{\rho(0)}.$$

Theorem 1.1. *Fix $\nu \geq 1$, and consider $-\Delta_\nu$ with the Dirichlet boundary condition. Assume either (i) or (ii) of the following conditions:*

- (i) $q_0 = 0$,
- (ii) $\nu = 1$ and $E_1 = 0$.

Then the mapping

$$\Psi : q \mapsto \left((\tilde{\mu}_n(q))_{n=1}^\infty, (\varkappa_n(q))_{n=1}^\infty \right)$$

defined by (1.23), (1.24) is a real-analytic isomorphism between \mathscr{W}_1^0 and $\mathcal{M}_1 \times \ell_1^2$, where \mathcal{M}_1 is defined by (1.21) with $\mu_n^0 = (\pi n)^2, n \geq 1$. In particular, in the symmetric case (the function q is odd and the manifold M is symmetric with respect to the plane $x = \frac{1}{2}$) the spectral data mapping

$$(1.26) \quad \tilde{\mu} : \mathscr{W}_1^{0, \text{odd}} \ni q \rightarrow (\tilde{\mu}_n)_1^\infty \in \mathcal{M}_1$$

is a real analytic isomorphism between $\mathscr{W}_1^{0, \text{odd}}$ and \mathcal{M}_1 .

1.4.2. *Mixed boundary condition.* We next consider $-\Delta_\nu, \nu \geq 1$, with mixed boundary condition:

$$(1.27) \quad \begin{cases} -\Delta_\nu f = -\frac{1}{\rho^2}(\rho^2 f')' + \frac{E_\nu}{r^2} f, & \varrho = r^{m/2} \quad \text{on } (0, 1), \\ f(0) = 0, \quad f'(1) + b f(1) = 0, & (b, q) \in \mathbb{R} \times \mathscr{W}_1^0. \end{cases}$$

Let $\mu_n = \mu_n(q, b)$, $n = 0, 1, 2, \dots$ be the associated eigenvalues. They satisfy

$$(1.28) \quad \begin{aligned} \mu_n(q, b) &= \mu_n^0 + c_0 + \tilde{\mu}_n(q, b), \\ \mu_n^0 &= \pi^2(n + \frac{1}{2})^2 + 2b, \\ (\tilde{\mu}_n)_1^\infty &\in \ell^2, \quad c_0 = \int_0^1 \left((q_0 + q)^2 + \frac{E_\nu}{r^2} \right) dx. \end{aligned}$$

where μ_n^0 's are the eigenvalues for the unperturbed case $r = 1$. As in [15], we introduce the norming constants

$$(1.29) \quad \chi_n(q, b) = \log \left| \frac{\varrho(1)f_n(1, q, b)}{f_n'(0, q, b)} \right|, \quad n \geq 0,$$

where f_n is the n -th eigenfunction satisfying $f_n'(0, q, b) \neq 0$ and $f_n(1, q, b) \neq 0$. When $q = b = 0$, a simple calculation gives

$$(1.30) \quad \chi_n^0 := \chi_n(0, 0) = -\log \pi(n + \frac{1}{2}).$$

Theorem 1.2. *For any fixed $(b, q_0, \nu) \in \mathbb{R}^2 \times \mathbb{N}$, consider $-\Delta_\nu$ with mixed boundary condition. Assume either (i) or (ii) of the following conditions:*

(i) $q_0 = 0$,

(ii) $\nu = 1$ and $E_1 = 0$.

Then the mapping defined by (1.28)-(1.30)

$$\Psi : q \mapsto ((\tilde{\mu}_n(q, b))_{n=1}^\infty, (\chi_{n-1}(q, b) - \chi_{n-1}^0)_{n=1}^\infty)$$

is a real-analytic isomorphism between \mathcal{W}_1^0 and $\mathcal{M}_1 \times \ell_1^2$, where \mathcal{M}_1 is defined by (1.21) with $\mu_n^0 = (\pi n + \frac{1}{2})^2 + 2b$, $n \geq 1$. Moreover, for each $(q, b) \in \mathcal{W}_1^0 \times \mathbb{R}$, the following identity holds:

$$(1.31) \quad b = \sum_{n=0}^\infty \left(2 - \frac{e^{\chi_n(q, b)}}{|\frac{\partial w}{\partial \lambda}(\mu_n, q, b)|} \right),$$

where the function $w(\lambda, q, b)$ is given by

$$(1.32) \quad w(\lambda, q, b) = \cos \sqrt{\lambda} \cdot \prod_{n=0}^\infty \frac{\lambda - \mu_n(q, b)}{\lambda - \mu_n^0}, \quad \lambda \in \mathbb{C}.$$

Here (1.31) and (1.32) converge uniformly on any bounded subsets in \mathbf{C} .

1.4.3. *Robin boundary conditions.* The 3rd case is the Robin boundary condition:

$$(1.33) \quad \begin{cases} -\Delta_\nu f = -\frac{1}{\rho^2}(\rho^2 f')' + \frac{E_\nu}{r^2}, & \varrho = r^{m/2} \quad \text{on } (0, 1), \\ f'(0) - af(0) = 0, & f'(1) + bf(1) = 0, \quad (a, b, q) \in \mathbb{R}^2 \times \mathcal{W}_1^0. \end{cases}$$

Let $\mu_n = \mu_n(q, a, b)$, $n = 0, 1, 2, \dots$ be the associated eigenvalues. It is well-known that

$$(1.34) \quad \begin{aligned} \mu_n &= \mu_n^0 + c_0 + \tilde{\mu}_n(q, a, b), \\ \mu_n^0 &= (n\pi)^2 + 2(a + b), \quad (\tilde{\mu}_n)_1^\infty \in \ell^2, \\ c_0 &= \int_0^1 \left((q_0 + q)^2 + \frac{E_\nu}{r^2} \right) dx. \end{aligned}$$

Note $\mu_n^0, n \geq 0$ are the eigenvalues for $r = 1$. The norming constants are defined by

$$(1.35) \quad \phi_n(q, a, b) = \log \left| \frac{\varrho(1)f_n(1, q, a, b)}{f_n(0, q, a, b)} \right|, \quad n \geq 0,$$

where f_n is the n -th eigenfunction. They satisfy $f_n(1, q, a, b) \neq 0$ and $f_n(0, q, a, b) \neq 0$.

Theorem 1.3. *For any fixed $(a, b, q_0, \nu) \in \mathbb{R}^3 \times \mathbb{N}$, consider $-\Delta_\nu$ with Robin boundary condition. Suppose either (i) or (ii) of the following conditions hold:*

(i) $q_0 = 0$,

(ii) $\nu = 1$ and $E_1 = 0$.

Then the mapping defined by (1.34), (1.35)

$$(1.36) \quad \Psi_{a,b} : q \mapsto \left((\tilde{\mu}_n(q, a, b))_{n=1}^\infty, (\phi_n(q, a, b))_{n=1}^\infty \right)$$

is a real-analytic isomorphism between \mathcal{W}_1^0 and $\mathcal{M}_1 \times \ell_1^2$, where \mathcal{M}_1 is defined by (1.21) with $\mu_n^0 = (\pi n)^2 + 2(a + b), n \geq 1$.

Remark. 1) In Theorems 1.1-1.3 we consider two cases: (i) $q_0 = 0$, or (ii) $\nu = 1$ and $E_1 = 0$. The inverse problems for the cases: (1) $q_0 \in \mathbb{R}, \nu \geq 2$ or (2) $\nu = 1$ and $E_1 \neq 0$ in Theorems 1.1-1.3 are still open.

2) We have the standard asymptotics (1.23), (1.28) and (1.33) for fixed ν . It is interesting to determine the asymptotics uniformly in $\nu \geq 1$.

1.5. Main results II. The curvature mapping. The Minkowski problem in classical differential geometry asks the existence of a convex surface with a prescribed Gaussian curvature. More precisely, for a given strictly positive real function F defined on a sphere, one seeks a strictly convex compact surface \mathcal{S} , whose Gaussian curvature at x is equal to $F(\mathbf{n}(x))$, where $\mathbf{n}(x)$ denotes the outer unit normal to \mathcal{S} at x . The Minkowski problem was solved by Pogorelov [20] and by Cheng-Yau [6].

We consider only the case $m = \dim Y = 1$. Note that our surface is not convex, in general. We solve an analogue of the Minkowski problem in the case of the surface of revolution by showing the existence of a bijection between the Gaussian curvatures and the profiles of surfaces.

As it is well-known, the Gaussian curvature \mathcal{K} is given by

$$(1.37) \quad \mathcal{K} = -\frac{r''}{r}, \quad \varrho = r^{\frac{1}{2}}.$$

As above, we represent the profile $r(x)$ in the following way:

$$(1.38) \quad r(x) = r_0 e^{2Q(x)}, \quad Q(x) = \int_0^x (q_0 + q(t)) dt, \quad (q_0, q) \in \mathbb{R} \times \mathcal{W}_1^0.$$

Then we have

$$(1.39) \quad \mathcal{K} = -2q' - 4(q_0 + q)^2.$$

Note that if $q = 0$, then $\mathcal{K} = -4q_0^2 < 0$ is a negative constant. Letting

$$(1.40) \quad \mathcal{K}_0 = 4 \int_0^1 (2q_0 q + q^2) dx, \quad G(q) = 2q' + 4(q_0 + q)^2 - \mathcal{K}_0,$$

we rewrite \mathcal{K} into the form

$$(1.41) \quad \mathcal{K} = -G(q) - \mathcal{K}_0 - 4q_0^2.$$

Theorem 1.4. *Let the Gaussian curvature \mathcal{K} and the profile $r(x)$ be given by (1.37), (1.38), where $(q_0, q) \in \mathbb{R} \times \mathcal{W}_1^0$. Then the mapping $G : \mathcal{W}_1^0 \rightarrow \mathcal{H}_0$ defined by*

$$(1.42) \quad q \rightarrow G(q) = -\mathcal{K} - \mathcal{K}_0 - 4q_0^2$$

is a real analytic isomorphism between \mathcal{W}_1^0 and \mathcal{H}_0 . Moreover, the constant \mathcal{K}_0 is uniquely defined by $G(q)$.

Remark. This theorem also holds with \mathcal{W}_1^0 replaced by \mathcal{H}_1 .

Theorem 1.4 gives the mapping between \mathcal{K} and the profile r . Thus Theorems 1.1 \sim 1.4 make the mapping

$$\text{Gaussian curvature } \mathcal{K} \rightarrow \text{eigenvalues} + \text{norming constants}$$

well-defined. We illustrate this by Theorem 1.5. We consider the Sturm-Liouville problem with Robin boundary condition:

$$(1.43) \quad -\frac{1}{\varrho^2}(\varrho^2 f')' + \frac{E_\nu}{r^2} f = \lambda f, \quad f'(0) - a f(0) = 0, \quad f'(1) + b f(1) = 0.$$

Let $\xi = G(q) \in \mathcal{H}_0$ and $A = (a, b, q_0) \in \mathbb{R}^3$. Let $\mu_n = \mu_n(\xi, A)$, $n = 0, 1, 2, \dots$ be the eigenvalues of (1.43). They satisfy

$$(1.44) \quad \begin{aligned} \mu_n(\xi, A) &= \mu_n^0 + c_0 + \tilde{\mu}_n(\xi, A), \\ \mu_n^0 &= (n\pi)^2 + 2(a + b), \quad (\tilde{\mu}_n)_1^\infty \in \ell^2, \\ c_0 &= \int_0^1 \left(q^2 + \frac{E_\nu}{r^2} \right) dx. \end{aligned}$$

Here μ_n^0 , $n \geq 0$, are the unperturbed eigenvalues for the case $r = 1$. We introduce the norming constants

$$(1.45) \quad \phi_n(\xi, A) = \log \left| \frac{\varrho(1) f_n(1, \xi, A)}{f_n(0, \xi, A)} \right|, \quad n \geq 1,$$

where f_n is the n -th eigenfunction. Note that $f_n(1, \xi, A) \neq 0$ and $f_n(0, \xi, A) \neq 0$.

Theorem 1.5. *Let $A = (a, b, q_0) \in \mathbb{R}^3$, $\nu \geq 1$ be fixed and consider $-\Delta_\nu$ with Robin boundary condition. Assume either (i) or (ii) of the following conditions:*

- (i) $q_0 = 0$,
- (ii) $\nu = 1$ and $E_1 = 0$.

Then the mapping defined by (1.44), (1.45)

$$(1.46) \quad \xi \rightarrow \Phi_A(\xi) = \left((\tilde{\mu}_n(\xi, A))_{n=1}^\infty, (\phi_n(\xi, A))_{n=1}^\infty \right)$$

is a real-analytic isomorphism between \mathcal{H}_0 and $\mathcal{M}_1 \times \ell_1^2$, where \mathcal{M}_1 is defined by (1.21) with $\mu_n^0 = (\pi n)^2 + 2(a+b)$, $n \geq 1$.

1.6. Brief overview. There is an abundance of works devoted to the spectral theory and inverse problems for the surface of revolution from the view points of classical inverse Sturm-Liouville theory, integrable systems, micro-local analysis, see [1], [7], [8], [9], [19] and references therein. Bruning-Heintz [5] proved that the symmetric metric is determined from the spectrum by using the 1-dimensional Gel'fand-Levitan theory [16], [18].

For integrable systems associated with surfaces of revolution, see e.g. [12], [24], [3], [22], [23] and references therein. Here we mention the work of Zelditch [25], which proved that the isospectral revolutionary surfaces of simple length spectrum, with some additional conditions, are isometric. In fact, the assumptions ensure the existence of global action-angle variables for the geodesic flow, which entails that the Laplacian has a global quantum normal form in terms of action operators. From the singularity expansion of the trace of wave group, one can then reconstruct the global quantum normal form, hence the metric. This argument, in due course, recovers the result of [5]. Note, however, that the class of metrics considered is shown to be residual in the class of metrics satisfying all the assumptions above concerning the metric but not the simple length spectrum assumption.

In the proof we use the analytic approach of Trubowitz and his co-authors (see [21] and references therein) plus its development for periodic systems [11]. Using them we obtain the global transformation for inverse Sturm-Liouville theory [10]. Note that for [10] the results of inverse Sturm-Liouville theory [21], [15] and [13] are important.

1.7. Plan of the paper. We start from proving Theorem 1.4, which is based on an abstract theorem in non-linear functional analysis [11]. In Section 2, we do it after preparing the estimates for the Riccati type mapping. The idea of the proof of Theorems 1.1, 1.2 and 1.3 consists in converting the Sturm-Liouville equation

$$-\frac{1}{\rho^2} (\rho^2 f')' + \frac{E}{r^2} = 0$$

to the Schrödinger equation

$$-y'' + py = Ey$$

using some non-linear mapping. In Section 3, we explain the results for the isomorphic property of the spectral data mapping. The paper [10] has been prepared for this purpose, and using the results there we shall prove Theorem 1.1, 1.2. and 1.3 in Section 4.

2. THE CURVATURE INVERSE PROBLEM AND RICCATI TYPE MAPPINGS

2.1. Estimates for Riccati type mappings. We define the mapping $G : \mathcal{H} \rightarrow \mathcal{H}_0$, where $\mathcal{H} = \mathcal{H}_1$ or $\mathcal{H} = \mathcal{W}_1^0$ by

$$(2.1) \quad \begin{cases} p = G(q) = q' + q^2 + 2q_0q - c_0, & c_0 = \int_0^1 (q^2 + 2q_0q)dx, \\ q_0 = \text{const} \in \mathbb{R}, & q \in \mathcal{W}_1^0 \quad \text{or} \quad q \in \mathcal{H}_1 \end{cases},$$

Lemma 2.1. *Let p be given by (2.1), where $q \in \mathcal{H}_1$ or $q \in \mathcal{W}_1^0$. Then the following estimates hold true:*

$$(2.2) \quad \|q'\|^2 \leq \|p\|^2 = \|q'\|^2 + \|q^2 + 2q_0q - c_0\|^2,$$

$$(2.3) \quad \|p\|^2 = \|q'\|^2 + \|q^2\|^2 + 4q_0^2\|q\|^2 + 4q_0(q^3, 1) - c_0^2,$$

$$(2.4) \quad \|p\|^2 \leq \|q'\|^2 + \|q^2\|^2 + 4q_0^2\|q\|^2 + 4q_0(q^3, 1),$$

where (\cdot, \cdot) is the scalar product in $L^2(0, 1)$.

Proof. Let $h = q^2 + 2q_0q - c_0$. We have

$$\begin{aligned} \|p\|^2 &= \|q'\|^2 + \|h\|^2 + 2(q', h), \\ (q', h) &= (q', q^2 + 2q_0q - c_0) = 0, \end{aligned}$$

where the integration by parts has been used. This yields (2.2). We have

$$\begin{aligned} \|h\|^2 &= \|q^2 + 2q_0q - c_0\|^2 = \|q^2 + 2q_0q\|^2 - 2(q^2 + 2q_0q, c_0) + c_0^2 \\ &= \|q^2 + 2q_0q\|^2 - c_0^2 = \|q^2\|^2 + 4q_0^2c_0 + 4q_0(q^2, q) - c_0^2, \\ \|q^2 + 2q_0q\|^2 &= \|q^2\|^2 + 4q_0^2\|q\|^2 + 4q_0(q^2, q) = \|q^2\|^2 + 4q_0^2\|q\|^2 + 4q_0(q^3, 1) \end{aligned}$$

and together with (2.2) this yields (2.3) and (2.4). ■

We show that that mapping $G = G(q) = G(q, q_0)$ in (2.1) is real analytic.

Lemma 2.2. *Let $\mathcal{H} = \mathcal{H}_1$ or $\mathcal{H} = \mathcal{W}_1^0$ and let $q_0 \in \mathbb{R}$. The mapping $G : \mathcal{H} \rightarrow \mathcal{H}_0$ given by (2.1) is real analytic and its gradient is given by*

$$(2.5) \quad \frac{\partial G(q)}{\partial q} f = f' + 2(q_0 + q)f - \int_0^1 2(q_0 + q)f dx, \quad \forall \quad q, f \in \mathcal{H}.$$

Moreover, the operator $\frac{\partial G(q)}{\partial q}$ is invertible for all $q \in \mathcal{H}$.

Proof. By the standard arguments (see [21]), we see that $G(q)$ is real analytic and its gradient is given by (2.5).

Due to (2.5), the linear operator $\frac{\partial G(q)}{\partial q} : \mathcal{H} \rightarrow \mathcal{H}_0$ is a sum of a boundedly invertible operator and a compact operator for all $q \in \mathcal{H}$. Hence $\frac{\partial G(q)}{\partial q}$ is a Fredholm operator. We prove that the operator $\frac{\partial G(q)}{\partial q}$ is invertible by contradiction. Let $f \in \mathcal{H}$ be a solution of the equation

$$(2.6) \quad \frac{\partial G(q)}{\partial q} f = 0, \quad f \neq 0,$$

for some fixed $q \in \mathcal{H}$. Due to (2.5) we have the equation

$$(2.7) \quad \frac{\partial G(q)}{\partial q} f = f' + 2(q_0 + q)f - C = 0, \quad C = \int_0^1 2(q_0 + q)f dx.$$

This implies

$$(2.8) \quad (e^{2Q}f)' = Ce^{2Q}, \quad Q = \int_0^x (q_0 + q)dt.$$

Let us first assume that the constant $C = 0$. Then we get $(e^{2Q}f)' = 0$, which yields $(e^{2Q}f)(x) = (e^{2Q}f)(0)$, $x \in [0, 1]$. If $f \in \mathcal{W}_1^0$, then we obtain $(e^{2Q}f) = 0$ and $f = 0$. If $f \in \mathcal{H}_1$, then we obtain $(e^{2Q}f)(x) = f(0)$ and $f = e^{-2Q}f(0)$. This gives $f = 0$, since $\int_0^1 f dt = 0$. In any case, we have arrived at a contradiction.

Next let us assume that $C \neq 0$. Without loss of generality, we can assume that $C = 1$. Then we get

$$(e^{2Q}f)(x) = f(0) + \int_0^x e^{-2Q} dt.$$

If $f \in \mathcal{W}_1^0$, then we obtain $(e^{2Q}f)(1) = \int_0^1 e^{-2Q} dt > 0$, which gives a contradiction.

If $f \in \mathcal{H}_1$, then we obtain $(e^{2Q}f)(1) = f(0) + \int_0^1 e^{-2Q} dt > f(0)$, which again gives a contradiction. Thus the operator $\frac{\partial G}{\partial q}$ is invertible for all $q \in \mathcal{H}$. ■

2.2. Analytic isomorphism. In order to prove Theorem 1.4 we use the "direct approach" in [11] based on nonlinear functional analysis. Our main tool is the following theorem in [11].

Theorem 2.3. *Let H, H_1 be real separable Hilbert spaces equipped with norms $\|\cdot\|, \|\cdot\|_1$. Suppose that the map $f : H \rightarrow H_1$ satisfies the following conditions:*

- i) *f is real analytic and the operator $\frac{d}{dq}f$ has an inverse for all $q \in H$,*
- ii) *there is a nondecreasing function $\eta : [0, \infty) \rightarrow [0, \infty)$, $\eta(0) = 0$, such that $\|q\| \leq \eta(\|f(q)\|_1)$ for all $q \in H$,*
- iii) *there exists a linear isomorphism $f_0 : H \rightarrow H_1$ such that the mapping $f - f_0 : H \rightarrow H_1$ is compact.*

Then f is a real analytic isomorphism between H and H_1 .

Proof of Theorem 1.4. We check all conditions in Theorem 2.3 for the mapping $\xi = G(q)$, $q \in \mathcal{W}_1^0$ given by (1.40). The proof for the case $q \in \mathcal{H}_1$ is similar. We rewrite this mapping in the form

$$\xi = G(v/2) = v' + 2v_0v + v^2 - c_0, \quad c_0 = \int_0^1 (2v_0v + v^2)dt,$$

where $v = 2q \in \mathcal{W}_1^0$ and $v_0 = 2q_0$ is a constant.

Lemma 2.2 implies the assertion (i), and Lemma 2.1 the assertion (ii). Let us check iii). We take a model mapping ξ_0 by $\xi_0(v) = v'$. Suppose $q^\nu \rightarrow q$ weakly in \mathcal{W}_1^0 as $\nu \rightarrow \infty$. Then $q^\nu \rightarrow q$ strongly in \mathcal{H}_0 as $\nu \rightarrow \infty$, since the imbedding mapping $\mathcal{W}_1^0 \rightarrow \mathcal{H}_0$ is compact. Hence the mapping $q \rightarrow \xi(v) - \xi_0(v)$ is compact.

Therefore, all conditions in Theorem 2.3 hold true and the mapping $G : \mathcal{W}_1^0 \rightarrow \mathcal{H}_0$ is a real analytic isomorphism between \mathcal{W}_1^0 and \mathcal{H}_0 . ■

3. SPECTRAL DATA MAPPING FOR THE CASE $\nu = 1$ AND $E_1 = 0$

3.1. Unitary transformations. Consider the Sturm-Liouville operator $-\Delta_q$ defined in $L^2((0, 1); \varrho^2 dx)$, where $\varrho = \varrho(x) > 0$, having the form

$$(3.1) \quad -\Delta_q f = -\frac{1}{\varrho^2}(\varrho^2 f')', \quad \varrho = r^{\frac{m}{2}} = e^Q,$$

equipped with the boundary condition

$$(3.2) \quad f'(0) - af(0) = 0, \quad f'(1) + bf(1) = 0, \quad a, b \in \mathbb{R} \cup \{\infty\}.$$

Here Q' is continuous on $[0, 1]$. We define the simple unitary transformation \mathcal{U} by

$$(3.3) \quad \mathcal{U} : L^2([0, 1], \varrho^2 dx) \rightarrow L^2([0, 1], dx), \quad \mathcal{U}f = \varrho f.$$

We transform the operator $-\Delta_q$ into the Schrödinger operator S_p by

$$(3.4) \quad \begin{aligned} \mathcal{U}(-\Delta_q)\mathcal{U}^{-1} &= -\varrho^{-1}\partial_x\varrho^2\partial_x\varrho^{-1} = \mathcal{D}^*\mathcal{D} = S_p + c_0, \quad S_p = -\frac{d^2}{dx^2} + p, \\ c_0 &= \int_0^1 (Q'' + (Q')^2)dx, \quad p = Q'' + (Q')^2 - c_0. \end{aligned}$$

since using the identity $\varrho = \varrho_0 e^Q$ we obtained

$$(3.5) \quad \begin{aligned} \mathcal{D} &= \varrho \partial_x \varrho^{-1} = \partial_x - Q', \quad \mathcal{D}^* = \left(\varrho \partial_x \varrho^{-1} \right)^* = -\partial_x - Q', \\ \mathcal{D}^*\mathcal{D} &= -(\partial_x + Q')(\partial_x - Q') = -\partial_x^2 + Q'' + (Q')^2. \end{aligned}$$

Here the operator $S_p = -\frac{d^2}{dx^2} + p$ acts in $L^2([0, 1], dx)$. We describe the boundary conditions for the operators $\Delta_q f$ and $S_p y$, where $y = \varrho f$. We have the following identities

$$(3.6) \quad \begin{aligned} y(0) &= f(0), \quad y'(0) = Q'(0)f(0) + f'(0), \\ y(1) &= \varrho(1)f(1), \quad y'(1) = Q'(1)\varrho(1)f(1) + \varrho(1)f'(1). \end{aligned}$$

The identities (3.6) yield the relations between the boundary conditions for f for Δ_q and y for S_p :

$$(3.7) \quad \begin{cases} f'(0) - af(0) = 0, \\ f'(1) + bf(1) = 0, \end{cases} \Leftrightarrow \begin{cases} y'(0) - (a + Q'(0))y(0) = 0, \\ y'(1) + (b - Q'(1))y(1) = 0, \end{cases} \quad a, b \in \mathbb{R} \cup \{\infty\}.$$

We consider the eigenvalue problems for $-\Delta_q$ and S_p on $(0, 1)$ subject to (3.7). Our second main theorem asserts that the above transformation $-\Delta_q \rightarrow S_p$ preserves the boundary conditions and spectral data.

Theorem 3.1. *Let $p = G(q)$, $q \in \mathcal{W}_1^0$, be defined by (2.1). Then the operators S_p and $-\Delta_q$, subject to the boundary condition (3.2), are unitarily equivalent. Moreover, they have the same eigenvalues and the norming constants.*

Proof. Let $p = G(q)$, $q \in \mathcal{W}_1^0$, be defined by (2.1). Then under the transformation $y = \mathcal{U}f = \varrho f$ the operators S_p and $-\Delta_q$ are unitarily equivalent. Moreover, due to $y = \varrho f$ and (3.2) the operators S_p and $-\Delta_q$ have the boundary conditions given by (3.7). Using (3.6) we can define the same norming constants. ■

Assume that the mapping $p \rightarrow (\text{eigenvalues} + \text{norming constants for the operator } S_p)$ gives the solution of the inverse problem for the operator S_p . Then, since the mapping $p \rightarrow q$ is an analytic isomorphism we obtain that the solution of the inverse problem for the mapping $p \rightarrow (\text{eigenvalues} + \text{norming constants for the operator } -\Delta_q)$.

Similar arguments work for the operator $-\Delta_q$ and the associated inverse problem. We will give a more precise explanation in the proof of Theorems 1.1 \sim 1.3.

Therefore, the inverse problem for $-\Delta_q$ is solvable if and only if so is for S_p . In this section, we consider the case $E_1 = 0$, $\nu = 1$.

3.2. Robin boundary condition. Consider the operator $-\Delta_q f = -\frac{1}{\varrho^2}(\varrho^2 f')'$ subject to the boundary condition (3.2) for the case $a, b \in \mathbb{R}$. We consider the case $q_0 \in \mathbb{R}$ and $E_1 = 0$. Let $A = (a, b, q_0) \in \mathbb{R}^3$. Let $\mu_n = \mu_n(q, A)$, $n \geq 0$, be the eigenvalues of Δ_q . Then we have

$$\mu_n(q, A) = \mu_n^0 + c_0 + \tilde{\mu}_n(q, A), \quad \text{where} \quad (\tilde{\mu}_n)_1^\infty \in \ell^2, \quad c_0 = \|q\|^2,$$

and $\mu_n^0 = (\pi n)^2 + 2(a+b)$, $n \geq 0$, denote the unperturbed eigenvalues. We introduce the norming constants

$$(3.8) \quad \phi_n(q, A) = \log \left| \frac{\varrho(1)f_n(1, q, A)}{f_n(0, q, A)} \right|, \quad n \geq 0,$$

where f_n is the n -th eigenfunction. Note that $f_n(1, q, A) \neq 0$ and $f_n(0, q, A) \neq 0$. The inverse problem for S_p with Robin boundary condition was solved in [15]. Therefore, applying Theorem 1.4 and the result of the inverse problem for S_p [15], we have the following theorem.

Theorem 3.2. *Let $E_1 = 0$ for $\nu = 1$. For each $A = (a, b, q_0) \in \mathbb{R}^3$, the mapping*

$$\Psi_A : q \mapsto ((\tilde{\mu}_n(q, A))_{n=1}^\infty; (\phi_n(q, A))_{n=1}^\infty)$$

is a real-analytic isomorphism between \mathcal{W}_1^0 and $\mathcal{M}_1 \times \ell_1^2$, where \mathcal{M}_1 is given by (1.21) with $\mu_n^0 = (\pi n)^2 + 2(a+b)$, $n \geq 1$.

Proof. Let $q \in \mathcal{W}_0^1$ and $A = (a, b, q_0) \in \mathbb{R}^3$. We consider the Sturm-Liouville problem with the generic boundary conditions,

$$-\frac{1}{\varrho^2}(\varrho^2 f')' = \lambda f, \quad f'(0) - af(0) = 0, \quad f'(1) + bf(1) = 0.$$

Let $\mu_n = \mu_n(q, A)$, $n = 0, 1, 2, \dots$ be the eigenvalues of the Sturm-Liouville problem. It is well known that

$$\mu_n(q, A) = \mu_n^0 + c_0 + \tilde{\mu}_n(q, A), \quad \text{where} \quad (\tilde{\mu}_n)_1^\infty \in \ell^2, \quad c_0 = \|q\|^2.$$

Following [15], we introduce the norming constants

$$(3.9) \quad \phi_n(q, A) = \log \left| \frac{\varrho(1)f_n(1, q, A)}{f_n(0, q, A)} \right|, \quad n \geq 0,$$

where f_n is the n -th eigenfunction. Thus for fixed $A \in \mathbb{R}^3$ we have the mapping

$$\Psi_A : q \mapsto \Psi_A(q) = ((\tilde{\mu}_n(q, A))_{n=1}^\infty; (\phi_n(q, A))_{n=1}^\infty)$$

Let $p = G(q), q \in \mathcal{W}_0^1$. We use Theorem 3.1. Consider the Sturm-Liouville problem

$$\begin{aligned} S_p y = -y'' + p(x)y, \quad y'(0) - a_0 y(0) = 0, \quad y'(1) + b_0 y(1) = 0, \\ a, b, q_0 \in \mathbb{R}, \quad a_0 = a - q_0, \quad b_0 = b + q_0. \end{aligned}$$

Denote by $\sigma_n = \sigma_n(p), n \geq 0$ the eigenvalues of S_p and let $\varkappa_n(p)$ be the corresponding norming constants given by

$$(3.10) \quad \varkappa_n(p) = \log \left| \frac{y_n(1, p, a_0, b_0)}{y'_n(0, p, a_0, b_0)} \right|, \quad n \geq 0.$$

Recall that due to [15] (see Proposition 5.4 in [15]) for each $a_0, b_0 \in \mathbb{R}$ the mapping

$$(3.11) \quad \Phi_{a_0, b_0} : p \mapsto \Phi_{a_0, b_0}(p) = ((\tilde{\sigma}_n(p))_{n=1}^\infty; (\varkappa_n(p))_{n=1}^\infty)$$

is a real-analytic isomorphism between \mathcal{H}_0 and $\mathcal{M}_1 \times \ell_1^2$.

Due to Theorem 1.4 we obtain the identity

$$\Phi_{a_0, b_0}(G(q)) = \Psi_A(q), \quad \forall q \in \mathcal{W}_0^1.$$

The mapping $\Psi_A(\cdot)$ is the composition of two mappings Φ_{a_0, b_0} and G , where each of them is the corresponding analytic isomorphism (see (3.11) and Theorem 1.4). Then for each $A \in \mathbb{R}^3$ the mapping

$$\Psi_A : q \mapsto ((\tilde{\mu}_n(q, A))_{n=1}^\infty; (\phi_n(q, A))_{n=1}^\infty)$$

is a real-analytic isomorphism between \mathcal{W}_0^1 and $\mathcal{M}_1 \times \ell_1^2$. ■

3.3. Dirichlet boundary condition. On the interval $[0, 1]$ we consider the operator $-\Delta_\nu = -\frac{1}{\varrho^2}(\varrho^2 f')'$ with Dirichlet boundary condition. We consider the case $\nu = 1, q_0 \in \mathbb{R}$ and $E_1 = 0$. Denote by $\mu_n = \mu_n(q), n = 1, 2, \dots$, the eigenvalues of $-\Delta_1$. It is well-known that all μ_n are simple and satisfy

$$(3.12) \quad \mu_n = \mu_n^0 + c_0 + \tilde{\mu}_n, \quad \mu_n^0 = (n\pi)^2, \quad (\tilde{\mu}_n)_1^\infty \in \ell^2, \quad c_0 = \int_0^1 (q_0 + q)^2 dx,$$

where $\mu_n^0 = (\pi n)^2, n \geq 1$, are the eigenvalues for the unperturbed case $r = 1$. We introduce the norming constants

$$(3.13) \quad \varkappa_n(q) = \log \left| \frac{\varrho(1)f'_n(1, q)}{f'_n(0, q)} \right|, \quad n \geq 1,$$

where f_n is the n -th eigenfunction of $-\Delta_\nu$. Note that $f'_n(0) \neq 0$ and $f'_n(1) \neq 0$. The inverse problem for S_p with the Dirichlet boundary condition was solved in [21]. Therefore, applying Theorem 1.4 and the result of the inverse problem for S_p [21], we have the following theorem.

Theorem 3.3. *Let $\nu = 1$ and $E_1 = 0$. For any $q_0 \in \mathbb{R}$ the mapping*

$$\Psi : q \mapsto ((\tilde{\mu}_n(q))_{n=1}^\infty; (\varkappa_n(q))_{n=1}^\infty)$$

is a real-analytic isomorphism between \mathcal{W}_1^0 and $\mathcal{M}_1 \times \ell_1^2$, where \mathcal{M}_1 is given by (1.21) with $\mu_n^0 = (\pi n)^2, n \geq 1$. In particular, in the symmetric case the spectral mapping

$$(3.14) \quad \tilde{\mu} : \mathcal{W}_0^{1, \text{odd}} \rightarrow \mathcal{M}_1, \quad \text{given by} \quad q \rightarrow \tilde{\mu}$$

is a real real analytic isomorphism between the Hilbert space $\mathcal{W}_0^{1, \text{odd}}$ and \mathcal{M}_1 .

Proof. The proof repeats the proof of Theorem 3.2, based on Theorem 3.1 and the well-known results from [21]. ■

3.4. Mixed boundary condition. We consider the operator $-\Delta_\nu = -\frac{1}{\rho^2}(\rho^2 f')'$ with mixed boundary condition $f(0) = 0, f'(1) + bf(1) = 0$, where $(b, q) \in \mathbb{R} \times \mathcal{W}_1^0$. We consider the case $\nu = 1, q_0 \in \mathbb{R}$ and $E_1 = 0$. Let $\mu_n = \mu_n(q, b), n = 0, 1, 2, \dots$ be the associated eigenvalues. They satisfy

$$(3.15) \quad \mu_n(q, b) = \mu_n^0 + c_0 + \tilde{\mu}_n(q, b), \quad (\tilde{\mu}_n)_1^\infty \in \ell^2, \quad c_0 = \int_0^1 (q_0 + q)^2 dx.$$

where $\mu_n^0 = \pi^2(n + \frac{1}{2})^2 + 2b$ are the eigenvalues for the unperturbed case $r = 1$. We introduce the norming constants

$$(3.16) \quad \chi_n(q, b) = \log \left| \frac{\varrho(1)f_n(1, q, b)}{f'_n(0, q, b)} \right|, \quad n \geq 0,$$

where f_n is the n -th eigenfunction satisfying $f'_n(0, q, b) \neq 0$ and $f_n(1, q, b) \neq 0$. When $q = b = 0$, a simple calculation gives $\chi_n^0 := \chi_n(0, 0) = -\log \pi(n + \frac{1}{2})$.

Theorem 3.4. *Let $\nu = 1$ and $E_1 = 0$ and let $b, q_0 \in \mathbb{R}$. Consider the inverse problem for (1.27) \sim (1.30) for any fixed $(b, q_0) \in \mathbb{R}^2$.*

(i) *The mapping*

$$\Psi : q \mapsto ((\tilde{\mu}_n(q, b))_{n=1}^\infty; (\chi_{n-1}(q, b) - \chi_{n-1}^0)_{n=1}^\infty)$$

is a real-analytic isomorphism between \mathcal{W}_1^0 and $\mathcal{M}_1 \times \ell_1^2$, where \mathcal{M}_1 is given by (1.21) with $\mu_n^0 = \pi^2(n + \frac{1}{2})^2 + 2b, n \geq 1$.

(ii) *For each $(q, b) \in \mathcal{W}_1^0 \times \mathbb{R}$ the following identity holds true:*

$$(3.17) \quad b = \sum_{n=0}^{+\infty} \left(2 - \frac{e^{\chi_n(q, b)}}{|\frac{\partial w}{\partial \lambda}(\mu_n, q, b)|} \right),$$

where the function $w(\lambda, q, b)$ is given by

$$(3.18) \quad w(\lambda, q, b) = \cos \sqrt{\lambda} \cdot \prod_{n=0}^{+\infty} \frac{\lambda - \mu_n(q, b)}{\lambda - \mu_n^0}, \quad \lambda \in \mathbb{C}.$$

Here both the product and the series converge uniformly on bounded subsets on the complex plane.

Proof. The proof is based on Theorem 3.1 and the results from [15]. We omit one, since it repeats the proof of Theorem 3.2. ■

3.5. Inverse problem for the curvature. We define the simple unitary transformation \mathcal{U} by

$$\mathcal{U} : L^2([0, 1], r dx) \rightarrow L^2([0, 1], dx), \quad y = \mathcal{U} f = r^{\frac{1}{2}} f, \quad \varrho = r^{\frac{1}{2}}.$$

Proof of Theorem 1.5. Consider the inverse problem for (1.43)-(1.45) for fixed $A = (a, b, q_0) \in \mathbb{R}^3$.

i) Let $q_0 = 0, \nu \geq 1$. We have two mappings $\xi = G(q)$ and

$$q \rightarrow \Psi_{A_0}(q) = \left((\tilde{\mu}_n(q, A_0))_{n=1}^\infty ; (\phi_n(q, A_0))_{n=1}^\infty \right)$$

and the composition of these mappings

$$(3.19) \quad \xi \rightarrow \Psi_{A_0}(G^{-1}(\xi)) = \Psi_{A_0} \circ G^{-1}(\xi)$$

Then due to Theorems 1.2 and 1.4, we deduce that the mapping $\Psi_{A_0} \circ G^{-1}$ is a real-analytic isomorphism between \mathcal{H}_0 and $\mathcal{M}_1 \times \ell_1^2$, where \mathcal{M}_1 is given by (1.21) with $\mu_n^0 = (\pi n)^2 + 2(a + b)$.

ii) Let $q_0 \in \mathbb{R}, \nu = 1, E_1 = 0$ and $(a, b, q) \in \mathbb{R}^2 \times \mathcal{W}_1^0$.

Consider the Sturm-Liouville operator $-\Delta_q$ given by

$$(3.20) \quad -\Delta_q f = -\frac{1}{\varrho^2}(\varrho^2 f')', \quad f'(0) - af(0) = 0, \quad f'(1) + bf(1) = 0.$$

Let $\mu_n = \mu_n(q, a, b), n = 0, 1, 2, \dots$ be the eigenvalues of the Sturm-Liouville problem (3.20). It is well known that

$$(3.21) \quad \mu_n = \mu_n^0 + c_0 + \tilde{\mu}_n(q, a, b), \quad \text{where } (\tilde{\mu}_n)_1^\infty \in \ell^2, \quad c_0 = \|q\|^2.$$

Here $\mu_n^0 = (\pi n)^2 + 2(a + b), n \geq 1$ are the unperturbed eigenvalues for $r = 1$. We introduce the norming constants

$$(3.22) \quad \phi_n(q, a, b) = \log \left| \frac{\varrho(1)f_n(1, q, a, b)}{f_n(0, q, a, b)} \right|, \quad n \geq 0,$$

where f_n is the n -th eigenfunction. Note that $f_n(1, a, q, b) \neq 0$ and $f_n(0, q, a, b) \neq 0$.

Under the transformation $\mathcal{U} : L^2([0, 1], \varrho^2 dx) \rightarrow L^2([0, 1], dx)$, given by $y = \mathcal{U} f = \varrho f$, we obtain

$$\mathcal{U}(-\Delta_{\varrho, u})\mathcal{U}^{-1} = S_p + c_0, \quad S_p y = -y'' + py,$$

where due to (3.6) the function y satisfies the following boundary conditions

$$(3.23) \quad \begin{cases} f'(0) - af(0) = 0, \\ f'(1) + bf(1) = 0, \end{cases} \Leftrightarrow \begin{cases} y'(0) - (a + q_0)y(0) = 0, \\ y'(1) + (b - q_0)y(1) = 0, \end{cases} \quad a, b \in \mathbb{R} \cup \{\infty\}.$$

We have two mappings $\xi = G(q)$ and

$$q \rightarrow \Psi_{a,b}(q) = \left((\tilde{\mu}_n(q, a, b))_{n=1}^\infty ; (\phi_n(q, a, b))_{n=1}^\infty \right)$$

and the composition of these mappings

$$(3.24) \quad \xi \rightarrow \Psi_{a,b}(G^{-1}(\xi)) = \Psi_{a,b} \circ G^{-1}(\xi)$$

Then due to Theorems 1.2 and 1.4, we deduce that the mapping $\Psi_{a,b} \circ G^{-1}$ is a real-analytic isomorphism between \mathcal{H}_0 and $\mathcal{M}_1 \times \ell_1^2$, where \mathcal{M}_1 is given by (1.21). \blacksquare

4. SPECTRAL DATA MAPPING FOR THE CASE $q_0 = 0$

4.1. Non-linear mapping. In view of (3.1), we take $\varrho(x)$ as follows

$$(4.1) \quad \varrho(x) = e^{Q(x)}, \quad Q = \int_0^x q(t)dt, \quad q_0 = 0.$$

We assume that the potential $u = u(Q)$ is related to q in the following way.

Condition U. *The function $u(\cdot)$ is real analytic and satisfies*

$$(4.2) \quad u'(t) \leq 0, \quad \forall t \in \mathbb{R}.$$

$$(4.3) \quad \|u'(Q)\| \leq F(\|q\|), \quad q \in \mathcal{W}_1^0,$$

for some increasing function $F : [0, \infty) \rightarrow [0, \infty)$. Here $\|\cdot\|$ denotes the norm of $L^2(0, 1)$.

Since ϱ, u are related with q by (4.1) and the Condition U, we write Δ_q instead of $\Delta_{\varrho, u}$. Now, we recall the theorem from [10] about the following mapping

$$(4.4) \quad p = P(q) = q' + q^2 + u(Q) - c_0, \quad c_0 = \int_0^1 (q' + q^2 + u(Q))dx.$$

Theorem 4.1. *The mapping $P : \mathcal{W}_1^0 \rightarrow \mathcal{H}_0$ given by (4.4) is a real analytic isomorphism between the Hilbert spaces \mathcal{W}_1^0 and \mathcal{H}_0 . In particular, the operator $\frac{\partial P}{\partial q}$ has an inverse for each $q \in \mathcal{W}_1^0$. Moreover, it has the following properties.*

(1) *Let $p = P(q), q \in \mathcal{W}_1^0$. Then the following estimates hold true*

$$(4.5) \quad \begin{aligned} \|q'\|^2 &\leq \|p\|^2 \leq \|q'\|^2 + 2\|q^2\|^2 + 2\|u\|^2 - c_0^2, \\ \|u\| &\leq \|q\|F(\|q\|). \end{aligned}$$

(2) *The mapping $P(q) - q' : \mathcal{W}_1^0 \rightarrow \mathcal{H}_0$ is compact.*

Furthermore, the mapping $q \rightarrow p = P(q), q \in \mathcal{W}_1^{0, \text{odd}}$ given by (4.4) is a real analytic isomorphism between the Hilbert spaces $\mathcal{W}_1^{0, \text{odd}}$ and $\mathcal{H}_0^{\text{even}}$.

Remark. 1) The mapping $q \rightarrow p = q' + q^2 + u - c_0 : \mathcal{H}_1 \rightarrow \mathcal{H}_0$ was considered in [13]. In some cases the mapping \mathcal{H}_0 into \mathcal{H}_{-1} is also useful (see [14], [2]).

2) In the case of inverse spectral theory for surfaces of revolution, we study the case of the function $u = E\varrho^{-\frac{4}{d}}$. Here $d+1 \geq 2$ is the dimension of the surface of revolution and $E \geq 0$ is a constant.

Our second main theorem asserts that the mapping in Theorem 4.1 preserves the boundary conditions and spectral data.

Theorem 4.2. *Let $p = P(q), q \in \mathcal{W}_1^0$, be defined by (4.4). Then the operators S_p and Δ_q are unitarily equivalent. In particular, they have the same boundary conditions, eigenvalues and the norming constants.*

Therefore, the inverse problem for Δ_q is solvable if and only if so is for S_p . Let us consider the following three cases separately.

4.2. Dirichlet boundary condition : $a = b = \infty$. Consider the Sturm-Liouville operator Δ_q defined in $L^2((0, 1); \varrho^2(x)dx)$, where $\varrho(x) = e^{Q(x)}$, having the form $\Delta_q f = -\frac{1}{\varrho^2}(\varrho^2 f')' + u(Q)f$ equipped with the boundary condition $f(0) = f(1) = 0$. Here $Q(x) = \int_0^x q(t)dt$, $q_0 = 0$ and u satisfies Condition U.

Denote by $\mu_n = \mu_n(q)$, $n \geq 1$, the eigenvalues of Δ_q subject to the boundary condition $f(0) = f(1) = 0$ for the case $a = b = \infty$. It is well-known that all μ_n are simple and satisfy

$$\mu_n = \mu_n^0 + c_0 + \tilde{\mu}_n, \quad \text{where} \quad (\tilde{\mu}_n)_1^{+\infty} \in \ell^2, \quad c_0 = \int_0^1 (q^2 + u)dt,$$

where $\mu_n^0 = (\pi n)^2$, $n \geq 1$, denote the unperturbed eigenvalues. The norming constants are defined by

$$(4.6) \quad \varkappa_n(q) = \log \left| \frac{\varrho(1)f'_n(1, q)}{f'_n(0, q)} \right|, \quad n \geq 1,$$

where f_n is the n -th eigenfunction. Note that $f'_n(0) \neq 0$ and $f'_n(1) \neq 0$. We recall theorem from [10].

Theorem 4.3. *Let $a = b = \infty$. Then the mapping*

$$\Psi : q \mapsto ((\tilde{\mu}_n(q))_{n=1}^\infty; (\varkappa_n(q))_{n=1}^\infty)$$

is a real-analytic isomorphism between \mathcal{H}_1^0 and $\mathcal{M}_1 \times \ell_1^2$, where \mathcal{M}_1 is defined by (1.21) with $\mu_n^0 = (\pi n)^2$, $n \geq 1$. In particular, in the symmetric case the spectral mapping

$$(4.7) \quad \tilde{\mu} : \mathcal{H}_1^{0, \text{odd}} \rightarrow \mathcal{M}_1, \quad \text{given by} \quad p \rightarrow \tilde{\mu}$$

is a real real analytic isomorphism between the Hilbert space $\mathcal{H}_1^{0, \text{odd}}$ and \mathcal{M}_1 .

4.3. Mixed boundary condition : $a = \infty, b \in \mathbb{R}$. Consider the Sturm-Liouville operator Δ_q defined in $L^2((0, 1); \varrho^2(x)dx)$, where $\varrho(x) = e^{Q(x)} > 0$, having the form $\Delta_q f = -\frac{1}{\varrho^2}(\varrho^2 f')' + u(Q)f$ equipped with the mixed boundary condition $f(0) = 0, f'(1) + bf(1) = 0$. Here $Q = \int_0^x q(t)dt$, $q_0 = 0$ and u satisfies Condition U.

Let $\mu_n = \mu_n(q, b)$, $n \geq 0$, be the eigenvalues of $-\Delta_q$ subject to the boundary condition $f(0) = 0, f'(1) + bf(1) = 0$ for the case $a = \infty, b \in \mathbb{R}$. We then have

$$\mu_n = \mu_n^0 + c_0 + \tilde{\mu}_n(q, b), \quad \text{where} \quad (\tilde{\mu}_n)_1^\infty \in \ell^2, \quad c_0 = \int_0^1 (q^2 + u)dt,$$

and $\mu_n^0 = \pi^2(n + \frac{1}{2})^2 + 2b$, $n \geq 0$, denote the unperturbed eigenvalues. The norming constants are defined by

$$(4.8) \quad \chi_n(q, b) = \log \left| \frac{\varrho(1)f_n(1, q, b)}{f'_n(0, q, b)} \right|, \quad n \geq 0,$$

where f_n is the n -th eigenfunction. Note that $f'_n(0, q, b) \neq 0$ and $f_n(1, q, b) \neq 0$. A simple calculation gives

$$\chi_n^0 = \chi_n(0, 0) = -\log \pi(n + \frac{1}{2}), \quad \text{where} \quad \sqrt{\mu_n^0} = \pi(n + \frac{1}{2}).$$

We recall theorem from [10].

Theorem 4.4. *i) For each fixed $b \in \mathbb{R}$ the mapping*

$$\Psi : q \mapsto ((\tilde{\mu}_n(q, b))_{n=1}^\infty; (\chi_{n-1}(q, b) - \chi_{n-1}^0)_{n=1}^\infty)$$

is a real-analytic isomorphism between \mathcal{W}_1^0 and $\mathcal{M}_1 \times \ell_1^2$, where \mathcal{M}_1 is defined by (1.21) with $\mu_n^0 = \pi^2(n + \frac{1}{2})^2 + 2b, n \geq 1$.

ii) For each $(q, b) \in \mathcal{W}_0^1 \times \mathbb{R}$ the following identity holds true:

$$(4.9) \quad b = \sum_{n=0}^{\infty} \left(2 - \frac{e^{\chi_n(q, b)}}{|\frac{\partial w}{\partial \lambda}(\mu_n, q, b)|} \right),$$

where

$$(4.10) \quad w(\lambda, q, b) = \cos \sqrt{\lambda} \cdot \prod_{n=0}^{+\infty} \frac{\lambda - \mu_n(q, b)}{\lambda - \mu_n^0}, \quad \lambda \in \mathbb{C}.$$

Here both the product and the series converge uniformly on bounded subsets on the complex plane.

4.4. Robin boundary condition : $a, b \in \mathbb{R}$. Consider the Sturm-Liouville operator Δ_q defined in $L^2((0, 1); \varrho^2(x)dx)$, where $\varrho(x) = e^{Q(x)} > 0$, having the form $\Delta_q f = -\frac{1}{\varrho^2}(\varrho^2 f')' + u(Q)f$ equipped with the generic boundary condition $f'(0) - af(0) = 0, f'(1) + bf(1) = 0$. Here $Q = \int_0^x q(t)dt$, $q_0 = 0$ and u satisfies Condition U.

Let $\mu_n = \mu_n(q, a, b), n \geq 0$, be the eigenvalues of Δ_q subject to the boundary condition $f'(0) - af(0) = 0, f'(1) + bf(1) = 0$ for the case $a, b \in \mathbb{R}$. Then we have

$$\mu_n = \mu_n^0 + c_0 + \tilde{\mu}_n(q, a, b), \quad \text{where} \quad (\tilde{\mu}_n)_1^\infty \in \ell^2, \quad c_0 = \int_0^1 (q^2 + u)dt,$$

and $\mu_n^0 = (\pi n)^2 + 2(a + b)$ denote the unperturbed eigenvalues. We introduce the norming constants

$$(4.11) \quad \phi_n(q, a, b) = \log \left| \frac{\varrho(1)f_n(1, q, a, b)}{f_n(0, q, a, b)} \right|, \quad n \geq 1,$$

where f_n is the n -th eigenfunction. Note that $f_n(1, a, q, b) \neq 0$ and $f_n(0, q, a, b) \neq 0$. We recall the results from [10].

Theorem 4.5. *For any $a, b \in \mathbb{R}$, the mapping*

$$\Psi_{a,b} : q \mapsto ((\tilde{\mu}_n(q, a, b))_{n=1}^\infty; (\phi_n(q, a, b))_{n=1}^\infty)$$

is a real-analytic isomorphism between \mathcal{W}_1^0 and $\mathcal{M}_1 \times \ell_1^2$, where \mathcal{M}_1 is given by (1.21) with $\mu_n^0 = (\pi n)^2 + 2(a + b)$.

4.5. Proof of Theorems 1.1 ~1.3. Recall that due to (1.2) we obtain that the Laplacian on (M, g) is unitarily equivalent to a direct sum of one-dimensional Schrödinger operators, namely, $-\Delta_{(M, g)} \simeq \oplus_{\nu \geq 1} \Delta_\nu$, where the direct sum acts in $\oplus_{\nu \geq 1} L^2([0, 1], dx)$. We consider the inverse problem for the operator Δ_ν for fixed $\nu \geq 1$ and $q_0 = 0$.

Proof of Theorem 1.1. We consider the inverse problem for the operator Δ_q given by

$$(4.12) \quad \begin{aligned} \Delta_\nu &= -\frac{1}{\varrho^2} \partial_x \varrho^2 \partial_x + \frac{E_\nu}{r^2}, \\ \varrho &= r^{\frac{m}{2}} = \varrho_0 e^Q, \quad Q(x) = \int_0^x (q_0 + q) dt, \quad q \in \mathcal{W}_1^0, \end{aligned}$$

under the Dirichlet boundary conditions $f(0) = f(1) = 0$ and for each $\nu \geq 1$.

Consider the case $q_0 = 0$. We apply Theorem 4.3 to our operator Δ_ν , since the function $u = \frac{E_\nu}{r^2} = E_\nu e^{-\frac{4}{m}Q}$ satisfies Condition U. Then Theorem 4.3 gives that the mapping $\Psi : q \mapsto ((\tilde{\mu}_n(q))_1^\infty; (\varkappa_n(q))_1^\infty)$ is a real-analytic isomorphism between \mathcal{W}_1^0 and $\mathcal{M}_1 \times \ell_1^2$, where \mathcal{M}_1 is given by (1.21) with $\mu_n^0 = (\pi n)^2$. In particular, in the symmetric case the spectral mapping $\tilde{\mu} : \mathcal{W}_0^{1, odd} \rightarrow \mathcal{M}_1$ given by $q \rightarrow \tilde{\mu}$ is a real analytic isomorphism between the Hilbert space $\mathcal{W}_0^{1, odd}$ and \mathcal{M}_1 .

The case $\nu = 1$ and $E_1 = 0$ has been considered in Theorem 3.3. ■

Proof of Theorem 1.2. We consider the inverse problem for the operator $-\Delta_\nu$ given by (4.12), under the mixed boundary conditions $f(0) = 0, f'(1) + bf(1) = 0$ for any fixed $(b, \nu) \in \mathbb{R} \times \mathbb{N}$.

Consider the case $q_0 = 0$. We apply Theorem 4.4 to our operator $-\Delta_\nu$, since the function $u = \frac{E_\nu}{r^2} = E_\nu e^{-\frac{4}{m}Q}$ satisfies Condition U. Then Theorem 4.4 gives that the mapping

$$\Psi : q \mapsto ((\tilde{\mu}_n(q, b))_{n=1}^\infty; (\chi_{n-1}(q, b) - \chi_{n-1}^0)_{n=1}^\infty)$$

is a real-analytic isomorphism between \mathcal{W}_1^0 and $\mathcal{M}_1 \times \ell_1^2$, where \mathcal{M}_1 is given by (1.21) with $\mu_n^0 = (\pi n + \frac{1}{2})^2 + 2b$. Moreover, for each $(q; b) \in \mathcal{W}_1^0 \times \mathbb{R}$ the following identity holds true:

$$(4.13) \quad b = \sum_{n=0}^{+\infty} \left(2 - \frac{e^{\chi_n(q, b)}}{|\frac{\partial w}{\partial \lambda}(\mu_n, q, b)|} \right),$$

where the function $w(\lambda, q, b)$ is given by

$$(4.14) \quad w(\lambda, q, b) = \cos \sqrt{\lambda} \cdot \prod_{n=0}^{+\infty} \frac{\lambda - \mu_n(q, b)}{\lambda - \mu_n^0}, \quad \lambda \in \mathbb{C}.$$

where both the product and the series converge uniformly on bounded subsets on the complex plane.

The case $\nu = 1$ and $E_1 = 0$ has been considered in Theorem 3.4. ■

Proof of Theorem 1.3. We consider the inverse problem for the operator $-\Delta_\nu$ given by (4.12), under the generic boundary conditions $f'(0) - af(0) = 0, f'(1) + bf(1) = 0$ for any fixed $(a, b, \nu) \in \mathbb{R}^2 \times \mathbb{N}$.

Consider the case $q_0 = 0$. We apply Theorem 4.5 to our operator $-\Delta_\nu$, since the function $u = \frac{E_\nu}{r^2} = E_\nu e^{-\frac{4}{m}Q}$ satisfies Condition U. Then Theorem 4.5 gives that the mapping

$$\Psi_{a,b} : q \mapsto ((\tilde{\mu}_n(q, a, b))_{n=1}^\infty ; (\phi_n(q, a, b))_{n=1}^\infty)$$

is a real-analytic isomorphism between \mathcal{W}_1^0 and $\mathcal{M}_1 \times \ell_1^2$, where \mathcal{M}_1 is given by (1.21) with $\mu_n^0 = (\pi n)^2 + 2(a + b)$.

The case $\nu = 1$ and $E_1 = 0$ has been considered in Theorem 3.2. ■

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REFERENCES

- [1] D. Abera, D and K. Agrawal, Surfaces of revolution in n dimensions. Internat. J. Math. Ed. Sci. Tech. 38 (2007), no. 6, 843–851.
- [2] A. Badanin, M. Klein and E. Korotyaev, The Marchenko-Ostrovski mapping and the trace formula for the Camassa-Holm equation. J. Funct. Anal. 203 (2003), no. 2, 494–518.
- [3] R. Beuther and B. G. Konopelchenko, Surface of revolution via the Schrödinger equation : Construction, integrable dynamics and visualization, Appl. Math. Comput. **101** (1999), 13–43.
- [4] P. M. Bleher, Distribution of energy levels of a quantum free particle on a surface of revolution. Duke Math. J. 74 (1994), no. 1, 45–93.
- [5] J. Brüning and E. Heintze, Spektrale Starrheit gewisser Drehflächen, Math. Ann. 269(1984), 95–101.
- [6] S.Y. Cheng and S.T. Yau, On the regularity of the solution of the n -dimensional Minkowski problem, Comm. Pure Appl. Math. 29(1976), 495–516.
- [7] M. Engman, Sharp bounds for eigenvalues and multiplicities on surfaces of revolution. Pacific J. Math. 186(1998), no. 1, 29–37.
- [8] J. Gravesen and M. Willatzen, Lew, L. C. Schrödinger problems for surfaces of revolution the finite cylinder as a test example. J. Math. Phys. 46 (2005), no. 1, 012107, 6 pp.
- [9] D. Gurarie, Semiclassical eigenvalues and shape problems on surfaces of revolution. J. Math. Phys. 36 (1995), no. 4, 1934–1944.
- [10] H. Isozaki and E. Korotyaev, Global transformations preserving Sturm-Liouville spectral data, will be published in RJM Phys.
- [11] P. Kargaev and E. Korotyaev, Inverse Problem for the Hill Operator, the Direct Approach. Invent. Math., 129 (1997), no. 3, 567–593.
- [12] B. G. Konopelchenko and I. A. Taimanov, Constant mean curvature surfaces via an integrable dynamical system. J. Phys. A 29 (1996), no. 6, 1261–1265.
- [13] E. Korotyaev, Invariance principle for inverse problems. Int. Math. Res. Not. 2002, no. 38, 2007–2020.
- [14] E. Korotyaev, Characterization of the spectrum of Schrödinger operators with periodic distributions. Int. Math. Res. Not. 2003, no. 37, 2019–2031.
- [15] E. L. Korotyaev and D. S. Chelkak, The inverse Sturm-Liouville problem with mixed boundary conditions. (Russian) Algebra i Analiz 21 (2009), no. 5, 114–137; translation in St. Petersburg Math. J. 21 (2010), no. 5, 761–778.
- [16] B. M. Levitan, *Inverse Sturm-Liouville problems* (Russian). Moscow: Nauka, 1984. English Translation: Utrecht: VNU Science Press, 1987.
- [17] C.T. Lin, Lower-bound estimates for eigenvalue of the Laplace operator on surfaces of revolution. Taiwanese J. Math. 7 (2003), no. 2, 20715.
- [18] Marchenko, V. A. Sturm-Liouville operator and applications, (Russian). Kiev: Naukova Dumka, 1977. English Translation: Basel: Birkhäuser, 1986.

- [19] R. F. Millar, The analytic continuation of solutions of the generalized axially symmetric Helmholtz equation. Arch. Rational Mech. Anal. 81(1983), no. 4, 349–372.
- [20] A. V. Pogorelov, The Minkowski multidimensional problem. (Russian) [Hilbert’s fourth problem] Izdat. “Nauka”, Moscow, 1974. 79 pp. English translation by V. Oliker. Introduction by Louis Nirenberg. Scripta Series in Mathematics. V. H. Winston & Sons, Washington, D.C.; Halsted Press [John Wiley & Sons], New York-Toronto-London, 1978.
- [21] J. Pöschel and E. Trubowitz, Inverse spectral theory. Pure and Applied Mathematics, 130. Academic Press, Inc., Boston, MA, 1987.
- [22] J. Sanders and J. Wang, Integrable systems in n -dimensional Riemannian geometry. Mosc. Math. J. 3 (2003), no. 4, 1369–1393.
- [23] M. Santoprete, Gravitational and harmonic oscillator potentials on surfaces of revolution. J. Math. Phys. 49(2008), no. 4, 042903, 16 pp.
- [24] I. Taimanov, Surfaces of revolution in terms of solitons. Ann. Global Anal. Geom. 15 (1997), no. 5, 419–35.
- [25] S. Zelditch, The inverse spectral problem for surfaces of revolution. J. Differential Geom. 49 (1998), no. 2, 207–264.

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